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J.A. Schouten and K. Yano

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MATHEMATICS

ON AN INTRINSIC CONNEXION IN AN X_{2n} WITH AN ALMOST HERMITIAN STRUCTURE

BY

J. A. SCHOUTEN AND K. YANO 1955

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In a previous paper [1]) one of us has derived three affine connexions in an almost Hermitian space that were characterized by the vanishing of $\nabla_j F_{ih}$ and by certain conditions imposed on the tensor $T_{ji}^{\cdot\cdot h} = \Gamma_{ji}^h - \overset{0}{\Gamma}{}_{ji}^h$ where the $\overset{0}{\Gamma}{}_{ji}^h$ are the parameters of the Riemannian connexion belonging to g_{ih}. Also a geometric interpretation of these connexions was given for the case where the space is complex. We prove in this paper that there is one and only one metric connexion satisfying $\nabla_j F_{ih} = 0$ and admitting infinitesimal parallelograms in the real invariant 2-directions. Also the geometric interpretations of the three former connexions will be given for the general case.

§ 1. *The X_{2n} with an almost complex structure*

If in an X_{2n} of class C^r; $r \geqslant 2$ a mixed tensorfield of valence 2 is given and if the components of this field with respect to a system (h) of real coordinates ξ^h; $h = 1, \ldots, 2n$, allowable in some neighbourhood, are real and of class C^{r-1} and satisfy the conditions

$$F_i^{\cdot j} F_j^{\cdot h} = - A_i^h \tag{1.1}$$

then the X_{2n} is said to possess an *almost complex structure*. Such an X_{2n} has only real points.

From (1.1) it follows that $F_i^{\cdot h}$ has n eigenvalues $+i$ and n eigenvalues $-i$ and that at each point there exists at least one local coordinate system with respect to which the matrix of F takes the diagonal form with n values $+i$ and n values $-i$ in the diagonal. But this means that in X_{2n} there must exist at least one (in general anholonomic) coordinate system (α); $\alpha = 1, \ldots, n, \bar{1}, \ldots, \bar{n}$ with basis vectors $\underset{\alpha}{e}{}^h$ and $\overset{\beta}{e}_i$ such that

$$F_i^{\cdot h} = i \overset{\varkappa}{e}_i \underset{\varkappa}{e}{}^h - i \overset{\bar\varkappa}{e}_i \underset{\bar\varkappa}{e}{}^h \quad ; \quad \varkappa = 1, \ldots, n \quad ; \quad \bar\varkappa = \bar{1}, \ldots, \bar{n}. \tag{1.2}$$

The basis vectors are non real vectors in the local E_{2n}. In this system the expression

$$\alpha)\quad (d\xi)^\alpha \overset{\text{def}}{=} A_h^\alpha \, d\xi^h; \qquad \beta)\quad A_h^\alpha \overset{\text{def}}{=} \underset{\beta}{e}{}^\alpha \overset{\beta}{e}_h \overset{*}{=} \overset{\alpha}{e}_h \tag{1.3}$$

is in general not a complete differential. The A_h^α are not real.

[1]) K. YANO, On three remarkable affine connexions in almost Hermitian spaces, Proc. Kon. Ned. Akad. Amsterdam, A 58, 24–32 (1955).

If we write [1])

$$B_i^h \overset{\text{def}}{=} \overset{\varkappa}{e}_i \underset{\varkappa}{e}^h \quad ; \quad C_i^h \overset{\text{def}}{=} \overset{\bar{\varkappa}}{e}_i \underset{\bar{\varkappa}}{e}^h \tag{1.4}$$

we have

$$B_j^h B_i^j = B_i^h \quad ; \quad B_j^h C_i^j = 0 \quad ; \quad C_j^h B_i^j = 0 \quad ; \quad C_j^h C_i^j = C_i^h \tag{1.5}$$

$$A_i^h = B_i^h + C_i^h \quad ; \quad F_i^{\cdot h} = iB_i^h - iC_i^h \tag{1.6}$$

and

$$\begin{cases} B_\lambda^\varkappa \overset{*}{=} \delta_\lambda^\varkappa \quad ; \quad B_\lambda^{\bar{\varkappa}} \overset{*}{=} 0 \quad ; \quad B_{\bar{\lambda}}^\varkappa \overset{*}{=} 0 \quad ; \quad B_{\bar{\lambda}}^{\bar{\varkappa}} \overset{*}{=} 0 \\ C_\lambda^\varkappa \overset{*}{=} 0 \quad ; \quad C_\lambda^{\bar{\varkappa}} \overset{*}{=} 0 \quad ; \quad C_{\bar{\lambda}}^\varkappa \overset{*}{=} 0 \quad ; \quad C_{\bar{\lambda}}^{\bar{\varkappa}} \overset{*}{=} \delta_{\bar{\lambda}}^{\bar{\varkappa}} \end{cases} \tag{1.7}$$

The $\underset{\lambda}{e}^h$ span a non real E_n in the local E_{2n} and the $\underset{\bar{\lambda}}{e}^h$ span another non real E_n that has no direction in common with the first. These E_n's are fixed by the tensor $F_i^{\cdot h}$ and conversely this tensor is determined uniquely by them [2]). We call them the *invariant* E_n's *of* $F_i^{\cdot h}$. Further there exists in every local E_{2n} a set of ∞^{2n-2} planes that are in an invariant way connected with $F_i^{\cdot h}$. Let u^h be an arbitrary real vector and let v^h be its transform by F, $v^h = u^i F_i^{\cdot h}$. Then $-u^h$ is the transform of v^h and the vectors u^h and v^h span such an invariant plane. We call these planes the *invariant* E_2's *of* $F_i^{\cdot h}$. At every point the infinitesimal affine transformation $A_i^h + F_i^{\cdot h} dt$ generates a 1-parameter group of real central affine transformations that leave all these planes invariant and the sections of each plane with the two invariant E_n's are the directions that are invariant for the transformations of this group.

We consider now the system of n homogeneous partial differential equations

$$\underset{\lambda}{e}^j \partial_j f(\xi^h) = 0 \quad ; \quad \lambda = 1, \ldots, n \tag{1.8}$$

or the adjoint system of n total differential equations

$$\overset{\bar{\varkappa}}{e}_j \, d\xi^j = 0 \quad ; \quad \bar{\varkappa} = \bar{1}, \ldots, \bar{n}. \tag{1.9}$$

In the case that the X_{2n} is of the class C^ω and that also the $F_i^{\cdot h}$ are of the class C^ω we can discuss the integrability of these systems, that can also be written in the forms

$$\alpha) \quad B_\lambda^j \partial_j f(\xi^h) = 0 \qquad \text{or} \qquad \beta) \quad A_\lambda^j \partial_j f(\xi^h) = 0 \tag{1.10}$$

$$\alpha) \quad C_j^{\bar{\lambda}} \, d\xi^j = 0 \qquad \text{or} \qquad \beta) \quad A_j^{\bar{\lambda}} \, d\xi^j = 0. \tag{1.11}$$

The integrability conditions of (1.10 β) or (1.11 β) are

$$\Omega_{\mu\lambda}^{\bar{\varkappa}} \overset{\text{def}}{=} A_\mu^j A_\lambda^i \partial_{[j} A_{i]}^{\bar{\varkappa}} = 0 \tag{1.12}$$

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1) Cf. J. A. Schouten, Ricci Calculus 1954 (here referred to as R. C.).

2) But for the sign.

where $\Omega^{\bar{\varkappa}}_{\mu\lambda}$ are some of the components $\Omega^{\alpha}_{\gamma\beta}$ of the object of anholonomity with respect to the coordinate system (α). If these conditions are satisfied the systems (1.8, 10) and (1.9, 11) are complete and there exist n independent solutions. In the same way the equations

$$(1.13)\qquad A^{j}_{\bar{\lambda}}\, \partial_j f(\xi^h) = 0$$

or

$$(1.14)\qquad A^{\lambda}_{j}\, d\xi^j = 0$$

form a complete system if and only if $\Omega^{\varkappa}_{\bar{\mu}\bar{\lambda}}=0$. If both conditions are satisfied and if we denote the solutions by $\xi^{\bar{\varkappa}'}$ and $\xi^{\varkappa'}$ respectively we get for the intermediate components of B and C

$$(1.15)\qquad \begin{cases} B^{j}_{\lambda}\, \partial_j\, \xi^{\bar{\varkappa}'} = 0 \quad ; \quad C^{j}_{\bar{\lambda}}\, \partial_j\, \xi^{\varkappa'} = 0 \\ C^{\bar{\lambda}}_{j}\, A^{j}_{\lambda'}\, d\xi^{\lambda'} = 0 \quad ; \quad B^{\lambda}_{j}\, A^{j}_{\bar{\lambda}'}\, d\xi^{\bar{\lambda}'} = 0. \end{cases}$$

or, introducing $\xi^{\varkappa'}$, $\xi^{\bar{\varkappa}'}$ as a holonomic coordinate system (α')

$$(1.16)\qquad \begin{cases} \underset{\lambda}{e^j}\, \overset{\bar{\varkappa}'}{e_j} = 0 \quad ; \quad \underset{\bar{\lambda}}{e^j}\, \overset{\varkappa'}{e_j} = 0 \\ \overset{\bar{\lambda}}{e_j}\, \underset{\lambda'}{e^j} = 0 \quad ; \quad \overset{\lambda}{e_j}\, \underset{\bar{\lambda}'}{e^j} = 0 \end{cases}$$

from which we see immediately that the $\underset{\lambda'}{e^h}$ $(\overset{\varkappa'}{e_i})$ depend only on the $\underset{\lambda}{e^h}(\overset{\varkappa}{e_i})$ and the $\underset{\bar{\lambda}'}{e^h}$ $(\overset{\bar{\varkappa}'}{e_i})$ only on the $\underset{\bar{\lambda}}{e^h}$ $(\overset{\bar{\varkappa}}{e_i})$. But this is also intuitively clear because the $\underset{\lambda'}{e^h}$ must span at every point one invariant E_n and the $\underset{\bar{\lambda}'}{e^h}$ the other E_n. For (α) we could now choose one of the holonomic coordinate systems just derived. Then all components $\Omega^{\alpha}_{\gamma\beta}$ would vanish. If we have two different sets of solutions and denote them by $\xi^{\varkappa}$, $\xi^{\bar{\varkappa}}$ and $\xi^{\varkappa'}$, $\xi^{\bar{\varkappa}'}$ we have

$$(1.17)\qquad \begin{cases} d\xi^{\varkappa} = B^{\varkappa}_{\varkappa'}\, d\xi^{\varkappa'} + B^{\varkappa}_{\bar{\varkappa}'}\, d\xi^{\bar{\varkappa}'} \quad ; \quad d\xi^{\bar{\varkappa}} = C^{\bar{\varkappa}}_{\varkappa'}\, d\xi^{\varkappa'} + C^{\bar{\varkappa}}_{\bar{\varkappa}'}\, d\xi^{\bar{\varkappa}'} \\ \phantom{d\xi^{\varkappa}} = B^{\varkappa}_{\varkappa'}\, d\xi^{\varkappa'} \qquad\qquad\qquad\qquad\quad = C^{\bar{\varkappa}}_{\bar{\varkappa}'}\, d\xi^{\bar{\varkappa}'} \end{cases}$$

because $B^{\varkappa}_{\bar{\varkappa}'}$ and $C^{\bar{\varkappa}}_{\varkappa}$ vanish. But this means that the $\xi^{\varkappa'}$ are analytic functions of the $\xi^{\varkappa}$ and that the $\xi^{\varkappa}$ are complex coordinates in an ordinary complex X_n from which the original X_{2n} is the auxiliary X_{2n} [1]). It is also said that the X_{2n} has a complex structure or that its structure is induced by an ordinary complex X_n. Hence, as was proved first by ECKMANN and FRÖLICHER [2]) an almost complex structure in X_{2n} of class C^{ω} is induced

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[1]) Cf. for an elaborate treatment of the quantities in an ordinary complex X_n and its auxiliary X_{2n}, R.C. VIII § 2–§ 8.

[2]) Sur l'intégrabilité des structures presque complexes, C.R. 232, 2284–2286 (1951).

by an ordinary X_n if and only if (1.10) (or 1.11) forms a complete system, that is if $\Omega^{\bar\varkappa}_{\mu\lambda}$ and $\Omega^{\varkappa}_{\bar\mu\bar\lambda}$ vanish.

In the ordinary complex X_n we have tensors of the first and of the second kind with all indices without a bar or all with a bar respectively. Also we have hybrid quantities, that are quantities with different kinds of indices. To an ordinary tensor for instance $P^{\varkappa}_{\cdot\lambda\mu}$ there belongs always the complex conjugate tensor of the second kind $P^{\bar\varkappa}_{\cdot\bar\lambda\bar\mu}$ and in the auxiliary X_{2n} these quantities form together a quantity with $2n^3$ components $P^{\varkappa}_{\cdot\lambda\mu}$, $P^{\bar\varkappa}_{\cdot\bar\lambda\bar\mu}$ with respect to (α). This quantity has the special property that at each point one part $P^{\varkappa}_{\cdot\lambda\mu}$ lies in one invariant E_n and the other part in the other E_n. But a general tensor of X_{2n} for instance $Q_{ji}^{\cdot\cdot h}$ has with respect to (α) the components $Q_{\mu\lambda}^{\cdot\cdot\varkappa}$, $Q_{\bar\mu\bar\lambda}^{\cdot\cdot\bar\varkappa}$, $Q_{\mu\lambda}^{\cdot\cdot\bar\varkappa}$, $Q_{\mu\bar\lambda}^{\cdot\cdot\varkappa}$, $Q_{\mu\bar\lambda}^{\cdot\cdot\bar\varkappa}$, $Q_{\bar\mu\bar\lambda}^{\cdot\cdot\varkappa}$, $Q_{\bar\mu\lambda}^{\cdot\cdot\varkappa}$, $Q_{\bar\mu\lambda}^{\cdot\cdot\bar\varkappa}$ and so it corresponds to one quantity of the first kind, one of the second kind and six hybrid quantities in X_n. If a quantity of X_{2n} lies wholly in the invariant E_n's we call it *pure* and in every other case *hybrid*. For instance A_i^h and $F_i^{\cdot h}$ are pure tensors. Of course every quantity can be split up into a pure and a hybrid part by using B_i^h and C_i^h. This is very useful, especially for the case of valence 2. Because

$$A_{ji}^{lk} = B_{ji}^{lk} + B_j^l\, C_i^k + C_j^l\, B_i^k + C_{ji}^{lk} \tag{1.18}$$

the operators

$$O_{ji}^{lk} = B_{ji}^{lk} + C_{ji}^{lk} \quad ; \quad {}'O_{ji}^{lk} = B_j^l\, C_i^k + C_j^l\, B_i^k \tag{1.19}$$

give the splitting up of every quantity of valence 2 and we have the formulae

$$\begin{cases} O = BB + CC \quad ; \quad {}'O = BC + CB \\ OO = O \quad ; \quad O{}'O = 0 \quad ; \quad {}'OO = 0 \quad ; \quad {}'O{}'O = {}'O \end{cases} \tag{1.20}$$

and

$$O = {}^1/_2\,(AA - FF) \quad ; \quad {}'O = {}^1/_2\,(AA + FF) \tag{1.21}$$

in an abridged notation. Pure quantities of a valence higher than 2 can be treated in the same way. For instance P_{jih} is pure if and only if it is invariant for transvection with $BBB + CCC$ and then transvection with every other combination, for instance BCB, gives zero and

$$P_{jih} = B_{j\,i\,h}^{\mu\lambda\varkappa}\, P_{\mu\lambda\varkappa} + C_{j\,i\,h}^{\bar\mu\bar\lambda\bar\varkappa}\, P_{\bar\mu\bar\lambda\bar\varkappa} \tag{1.22}$$

and for instance

$$P_{jik}\, B_h^k = B_{j\,i\,h}^{\mu\lambda\varkappa}\, P_{\mu\lambda\varkappa} \quad ; \quad P_{\mu\lambda\varkappa} = B_{\mu\lambda\varkappa}^{j\,i\,h}\, P_{jih}. \tag{1.23}$$

The condition of integrability can be put in another form. From (1.12) we get

$$\begin{cases} \Omega^{\bar\varkappa}_{\mu\lambda} = B^{j\,i}_{\mu\lambda}\, \partial_{[j}\, C^{\bar\varkappa}_{i]} = B^{j\,i}_{\mu\lambda}\, C^{\bar\varkappa}_h\, \partial_{[j}\, C^h_{i]} = \\ \quad = {}^1/_2\, i\, B^{j\,i}_{\mu\lambda}\, C^{\bar\varkappa}_h\, \partial_{[j}\, F_{i]}^{\cdot h}; \end{cases} \tag{1.24a}$$

(1.24b) $$\Omega_{\bar{\mu}\bar{\lambda}}^{\varkappa} = -\,{}^1/_2\, i\, C_{\bar{\mu}\bar{\lambda}}^{ji}\, B_h^{\varkappa}\, \partial_{[j}\, F_{i]}^{\cdot h}$$

or

(1.25a) $$\Omega_{\mu\lambda}^{\bar{\varkappa}}\, C_{\bar{\varkappa}}^{h} = {}^1/_2\, i\, B_{\mu\lambda}^{ji}\, \partial_{[j}\, F_{i]}^{\cdot h};$$

(1.25b) $$\Omega_{\bar{\mu}\bar{\lambda}}^{\varkappa}\, B_{\varkappa}^{h} = -\,{}^1/_2\, i\, C_{\bar{\mu}\bar{\lambda}}^{ji}\, \partial_{[j}\, F_{i]}^{\cdot h}.$$

Now, if $P_i^{\cdot h}$ is any tensor, it was proved by NIJENHUIS[1]) that the expression

(1.26) $$2\, P_{[j}^{\cdot l}\, (\partial_{|l|}\, P_{i]}^{\cdot h} - \partial_{i]}\, P_l^{\cdot h})$$

is also a tensor. If we denote the Nijenhuis tensor of $F_i^{\cdot h}$ by $N_{ji}^{\cdot\cdot h}$ we get

(1.27) $$N_{ji}^{\cdot\cdot k}\, F_k^{\cdot h} = 4\, O_{ji}^{lk}\, \partial_{[l}\, F_{k]}^{\cdot h}$$

and this proves that $N_{ji}^{\cdot\cdot h}$ is pure in the indices ji. From (1.27) we get

(1.28) $$N_{ji}^{\cdot\cdot h} = 8\, \Omega_{\mu\lambda}^{\bar{\varkappa}}\, B_{ji}^{\mu\lambda}\, C_{\bar{\varkappa}}^{h} + 8\, \Omega_{\bar{\mu}\bar{\lambda}}^{\varkappa}\, C_{ji}^{\bar{\mu}\bar{\lambda}}\, B_{\varkappa}^{h}$$

or in another form

(1.29) $$\begin{cases} \alpha) & N_{\mu\lambda}^{\cdot\cdot\varkappa} \overset{*}{=} 0; \quad \text{conj.} \\ \beta) & N_{\bar{\mu}\lambda}^{\cdot\cdot\varkappa} \overset{*}{=} 0; \quad \text{conj.} \\ \gamma) & N_{\mu\lambda}^{\cdot\cdot\bar{\varkappa}} \overset{*}{=} 8\, \Omega_{\mu\lambda}^{\bar{\varkappa}}; \quad \text{conj.} \end{cases}$$

from which we see that $N_{ji}^{\cdot\cdot h}$ is hybrid in the indices ih. As the only non vanishing components of $N_{ji}^{\cdot\cdot h}$ are equal to $8\Omega_{\mu\lambda}^{\bar{\varkappa}}$ and $8\Omega_{\bar{\mu}\bar{\lambda}}^{\varkappa}$ we get the theorem of ECKMANN and FRÖLICHER in the form:

An almost complex X_{2n} of class C^ω is complex if and only if the Nijenhuis tensor of $F_i^{\cdot h}$ vanishes.

In the analytic case the vanishing of $N_{ji}^{\cdot\cdot h}$ means that the two E_n-fields are building (complex) X_n's. But in the general case such an interpretation is impossible because there are no complex points in X_{2n} and therefore only real directions. In order to find a geometric interpretation in the general case we consider a vectorfield w_i and its rotation $w_{ji} = 2\partial_{[j}\, w_{i]}$. Let $u_i = F_i^{\cdot h} w_h$ be its transform by F and u_{ji} the rotation of u_i. Then it is easy to prove that

(1.30) $$2\, O w_{ji} = N_{ji}^{\cdot\cdot h}\, w_h + 2 F_j^{\cdot h}\, O u_{ih}$$

and this proves:

A vectorfield with a hybrid rotation is always transformed by F into a vectorfield whose rotation is also hybrid, if and only if $N_{ji}^{\cdot\cdot h} = 0$

and

$N_{ji}^{\cdot\cdot h}$ vanishes if and only if there exist m vectorfields such that their rotations and the rotations of their transforms by F are hybrid and if among these 2m vectorfields there are 2n linearly independent ones.

[1]) X_{n-1}-forming sets of eigenvectors, Proc. Kon. Ned. Akad. v. Wet. Amsterdam, **A 54**, 200–212 (1951).

According to the first theorem a gradientfield must have a transform with a hybrid rotation. But this rotation need not be zero.

§ 2. *An intrinsic connexion in an almost Hermitian space*

An almost complex X_{2n} is said to be almost Hermitian if a symmetric tensor g_{ih} of rank $2n$ and class C^{r-1} is introduced satisfying the condition

$$F_i^{\cdot k} F_h^{\cdot j} g_{kj} = g_{ih}. \tag{2.1}$$

Using g_{ih} for raising and lowering of indices we see that F_{ih} is a bivector. A general bivector in a $2n$-dimensional flat metric space can be split up into n blades[1]) and these blades are uniquely determined if the eigenvalues of the bivector are all different. But in the case of F_{ih} the blades are undetermined and for each of the invariant E_2's mentioned in § 1 the splitting up can be done in such a way that one blade just lies in this E_2. By g_{ih} each E_2 is now a real R_2 and the invariant E_n's are lying on the null-cone of g_{ih} and they cut each invariant R_2 in its null directions. The transformations of the 1-parameter group mentioned in § 1 are now rotations.

We wish to establish now a connexion satisfying the conditions

a. the connexion is metric:

$$\nabla_j g_{ih} = 0 \tag{2.2}$$

b. the two E_n-fields are invariant:

$$\nabla_j F_{ih} = 0 \tag{2.3}$$

c. in each invariant R_2 there exist infinitesimal parallelograms.

First we use the anholonomic coordinatesystem (α). We remark that g_{ih} and F_{ih} are hybrid and that (stars being dropped)

$$\begin{cases} F_{\lambda\bar{\varkappa}} = i\, g_{\lambda\bar{\varkappa}}; & \text{conj.} \\ F_{\lambda\varkappa} = 0; & \text{conj.} \end{cases} \tag{2.4}$$

From this we see that N_{jih} is pure in all indices because

$$\begin{cases} N_{\mu\lambda\bar{\varkappa}} = 0; & \text{conj.} \\ N_{\mu\lambda\varkappa} = 8\, \Omega_{\mu\lambda}^{\bar{\varkappa}}\, g_{\varkappa\bar{\varkappa}}; & \text{conj.} \\ N_{\mu\bar{\lambda}\bar{\varkappa}} = 0; & \text{conj.} \end{cases} \tag{2.5}$$

The second condition expresses that every direction in a local E_n has to remain in the local E_n after parallel displacement, hence this condition is equivalent with

$$\alpha)\quad \boxed{\Gamma_{\mu\lambda}^{\bar{\varkappa}} = 0}\,; \qquad \beta)\quad \boxed{\Gamma_{\mu\bar{\lambda}}^{\varkappa} = 0}\,; \quad \text{conj.} \tag{2.6}$$

The third condition is equivalent with

$$S_{\mu\bar{\lambda}}^{\cdot\cdot\varkappa} = 0; \qquad S_{\bar{\mu}\lambda}^{\cdot\cdot\varkappa} = 0; \quad \text{conj.} \tag{2,7}$$

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[1]) Cf. R.C., p. 46.

but because of the well known formula [1])

(2.8) $$S_{\gamma\beta}^{\cdot\cdot\alpha} = \Gamma_{[\gamma\beta]}^{\alpha} + \Omega_{\gamma\beta}^{\alpha}; \quad \text{conj.}$$

this leads to

(2.9) $$\Gamma_{\bar{\mu}\lambda}^{\varkappa} = -\,2\,\Omega_{\bar{\mu}\lambda}^{\varkappa}; \quad \text{conj.}$$

The first condition leads to the well known formula [2])

(2.10) $$\Gamma_{\gamma\beta}^{\alpha} = \overset{0}{\Gamma}{}_{\gamma\beta}^{\alpha} + S_{\gamma\beta}^{\cdot\cdot\alpha} - S_{\gamma.\beta}^{\cdot\alpha} - S_{\beta.\gamma}^{\cdot\alpha}; \quad \text{conj.}$$

and writing out this equation we get

(2.11) $$\begin{cases} \alpha) & \Gamma_{\mu\lambda}^{\varkappa} = \overset{0}{\Gamma}{}_{\mu\lambda}^{\varkappa} + S_{\mu\lambda}^{\cdot\cdot\varkappa}; \quad \text{conj.} \\ \beta) & 0 = \Gamma_{\mu\bar{\lambda}}^{\varkappa} = \overset{0}{\Gamma}{}_{\mu\bar{\lambda}}^{\varkappa} - S_{\bar{\lambda}\bar{\varkappa}}^{\cdot\cdot\bar{\mu}}\, g^{\bar{\varkappa}\varkappa}\, g_{\mu\bar{\mu}}; \quad \text{conj.} \\ \gamma) & -2\,\Omega_{\bar{\mu}\lambda}^{\varkappa} = \Gamma_{\bar{\mu}\lambda}^{\varkappa} = \overset{0}{\Gamma}{}_{\bar{\mu}\lambda}^{\varkappa} - S_{\bar{\mu}\bar{\varkappa}}^{\cdot\cdot\bar{\lambda}}\, g^{\bar{\varkappa}\varkappa}\, g_{\lambda\bar{\lambda}}; \quad \text{conj.} \\ \delta) & 0 = \Gamma_{\mu\lambda}^{\bar{\varkappa}} = \overset{0}{\Gamma}{}_{\mu\lambda}^{\bar{\varkappa}} + (S_{\mu\lambda\varkappa} - S_{\mu\varkappa\lambda} - S_{\lambda\varkappa\mu})\, g^{\bar{\varkappa}\varkappa}; \quad \text{conj.} \end{cases}$$

From these equations, (2.11 β) and (2.11 γ) are equivalent because of (2.8), and from them we derive

2.13) $$S_{\mu\lambda}^{\cdot\cdot\varkappa} = \overset{0}{\Gamma}{}_{\bar{\varkappa}\mu}^{\bar{\lambda}}\, g^{\bar{\varkappa}\varkappa}\, g_{\lambda\bar{\lambda}}; \quad \text{conj.}$$

Hence, in combination with (2.11 α)

(2.14) $$\boxed{\Gamma_{\mu\lambda}^{\varkappa} = \overset{0}{\Gamma}{}_{\mu\lambda}^{\varkappa} + \overset{0}{\Gamma}{}_{\bar{\varkappa}\mu}^{\bar{\lambda}}\, g^{\bar{\varkappa}\varkappa}\, g_{\lambda\bar{\lambda}}}\ ; \quad \text{conj.}$$

The connexion is now determined by (2.6) and (2.14).

From (2.11 δ) we get

(2.15) $$S_{\mu\lambda}^{\cdot\cdot\bar{\varkappa}} = -\,\overset{0}{\Gamma}{}_{[\mu\lambda]}^{\bar{\varkappa}}; \quad \text{conj.}$$

but from (1.29 γ) and (2.8) it follows that

(2.16) $$S_{\mu\lambda}^{\cdot\cdot\bar{\varkappa}} = \Omega_{\mu\lambda}^{\bar{\varkappa}} = {}^{1}\!/_{8}\, N_{\mu\lambda}^{\cdot\cdot\bar{\varkappa}}; \quad \text{conj.}$$

because the Riemannian connexion is symmetric. Hence

(2.17) $$\overset{0}{\Gamma}{}_{[\mu\lambda]}^{\bar{\varkappa}} = -\,\Omega_{\mu\lambda}^{\bar{\varkappa}} \quad {}^{3)}; \quad \text{conj.}$$

Now using the holonomic coordinates (h) we get from the first condition for $T_{ji}^{\cdot\cdot h} = \Gamma_{ji}^{h} - \overset{0}{\Gamma}{}_{ji}^{h}$ the equivalent

(2.18) $$T_{jih} = S_{jih} - S_{jhi} - S_{ihj}.$$

[1]) Cf. for instance R.C., p. 169.

[2]) Cf. for instance R.C., p. 132. $\overset{0}{\Gamma}{}_{\gamma\beta}^{\alpha}$ = param. Riem. connexion.

[3]) This is in accordance with R.C. (3.6 d) on p. 396 and (9.8) on p. 170.

The second condition is equivalent to

$$(2.19)\qquad \begin{cases} 0 = \nabla_j F_{ih} = \overset{0}{\nabla}_j F_{ih} - (S_{jik} - S_{jki} - S_{ikj}) F^{k}_{\cdot h} - \\ \qquad - (S_{jhk} - S_{jkh} - S_{hkj}) F_i^{\cdot k} \end{cases}$$

hence

$$(2.20)\qquad \begin{cases} 0 = F_l^{\cdot h} \nabla_j F_{ih} = F_l^{\cdot h} \overset{0}{\nabla}_j F_{ih} - 2\, O_{il}^{kh} S_{jkh} + \\ \qquad + 2\, O_{li}^{hk} S_{jhk} + 2\, O_{il}^{kh} S_{khj}. \end{cases}$$

The third condition is satisfied if and only if $S_{ji}^{\cdot\cdot h}$ is pure in ji because in every invariant E_2 there lies one vector of each invariant E_n. Hence

$$(2.21)\qquad O_{il}^{kh} S_{khj} = S_{ilj}.$$

Now the expression (1.26) for $N_{ji}^{\cdot\cdot h}$ can also be written in the form

$$(2.22)\qquad N_{jih} = 2\, F_{[j}^{\cdot l} (\overset{0}{\nabla}_{|l|} F_{i]h} - \overset{0}{\nabla}_{i]} F_{lh})$$

and from this follows the identity

$$(2.23)\qquad N_{jih} = 8\, O_{ih}^{ml} S_{jml}.$$

Hence from (2.20) and (2.23) and from the fact that N_{jih} is pure we get

$$(2.24)\qquad S_{jih} = {}^1\!/_2 (\overset{0}{\nabla}_h F_{lj}) F_i^{\cdot l} - {}^1\!/_4 N_{h[ij]}$$

and consequently

$$(2.25)\qquad \boxed{\begin{aligned} T_{jih} = {}^1\!/_2 (\overset{0}{\nabla}_h F_{lj}) F_i^{\cdot l} - {}^1\!/_2 (\overset{0}{\nabla}_i F_{lj}) F_h^{\cdot l} - \\ - {}^1\!/_2 (\overset{0}{\nabla}_j F_{li}) F_h^{\cdot l} - {}^1\!/_4 N_{hij} \end{aligned}}$$

by which equation the connexion is determined. For the analytic case with $N_{ji}^{\cdot\cdot h} = 0$ the connexion is of course identical with the unitary connexion in a $\tilde{U}_n$ [1]). For the pseudo-Kählerian case it is Riemannian.

§ 3. *On the geometric interpretation of the three other connexions*

1. Take an arbitrary contravariant vector u^h. We can associate with this contravariant vector a covariant vector defined by $F_{ih} u^h$. The hyperplane representing the covariant vector $F_{ih} u^h$ contains the direction representing the contravariant vector u^h. So this is a so-called null-system.

Now we transport the contravariant vector u^h parallelly with respect to the Riemannian connexion from the point ξ^h to the point $\xi^h + d\xi^h$. Then we get at $\xi^h + d\xi^h$

$$(3.1)\qquad u^h - d\xi^j \left\{ {h \atop ji} \right\} u^i.$$

[1]) R.C. p. 395 ff.

Next we transport the covariant vector $F_{ih}u^h$ parallelly with respect to the affine connexion Γ^h_{ji} from the point ξ^h to $\xi^h+d\xi^h$. Then we get at $\xi^h+d\xi^h$

$$F_{ih}\,u^h - d\xi^j\,\Gamma^l_{ji}\,F_{lh}\,u^h$$

or

$$F_{ih}\,u^h - d\xi^j\left(\left\{ {l \atop j\,i} \right\} + T_{ji}^{\cdot\cdot l}\right) F_{lh}\,u^h. \tag{3.2}$$

We assume that the hyperplane representing (3.2) contains the direction representing (3.1) for any vector u^h and for any displacement $d\xi^h$. This condition can be expressed as

$$T_{ji}^{\cdot\cdot l}\,F_{lh} + T_{jh}^{\cdot\cdot l}\,F_{li} = 0. \tag{3.3}$$

This corresponds to the condition which was used to get the first connexion in the previous paper.

2. Consider again a contravariant vector u^h and a covariant vector $F_{ih}u^h$ which contains the vector u^h. We transport the covariant vector $F_{ih}\,u^h$ parallelly with respect to the Riemannian connexion, and with respect to the affine connexion Γ^h_{ji} respectively from the point ξ^h to the point $\xi^h+u^h\varepsilon$ where ε is an infinitesimal.

Then we get at $\xi^h+u^h\varepsilon$

$$F_{ih}\,u^h - u^j\,\varepsilon\left\{ {l \atop j\,i} \right\} F_{lh}\,u^h$$

and

$$F_{ih}\,u^h - u^j\,\varepsilon\left[\left\{ {l \atop j\,i} \right\} + T_{ji}^{\cdot\cdot l}\right] F_{lh}\,u^h.$$

We assume that these two vectors coincide for any vector u^h. This condition can be expressed as

$$T_{ji}^{\cdot\cdot l}\,F_{lh} + T_{hi}^{\cdot\cdot l}\,F_{lj} = 0. \tag{3.4}$$

This corresponds to the condition which was used to get the second connexion in the previous paper.

3. Now, if we assume that the autoparallel curves with respect to Γ^h_{ji} coincides with the geodesics of the Riemannian metric $ds^2=g_{ih}d\xi^i d\xi^h$, then we have

$$\Gamma^h_{(ji)} = \left\{ {h \atop j\,i} \right\},$$

which can be expressed by $T_{(ji)}^{\cdot\cdot h}=0$ or by

$$T_{ji}^{\cdot\cdot l}\,F_{lh} + T_{ij}^{\cdot\cdot l}\,F_{lh} = 0. \tag{3.5}$$

This corresponds to the condition which was used to get the third connexion in the previous paper.